CERTAIN MAGNETOHYDRODYNAMIC FLOWS ASSOCIATED WITH THE PROPAGATION OF WAVES

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The propagation of waves in an absorptive medium is accompanied by unidirectional motion of this medium – a flow which develops due to the fact that the wave loses a portion of its momentum along with energy loss. It is this loss which is compensated by the flow by virtue of the momentum-conservation law. In a conducting medium the momentum losses via the waves are associated not only with the viscosity and thermal conductivity but also with the Joule-heat losses. Moreover, the magnetic field itself also affects the configuration and character of the flow in this case.

The motion of a conducting fluid in a magnetic field can be described, as is well known, by the following system of equations [1]:

$$\operatorname{div}\mathbf{H}=0 \qquad \frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0 \qquad \frac{\partial\mathbf{H}}{\partial t}=\operatorname{rot}(\mathbf{v}\times\mathbf{H})+\frac{c^2}{4\pi\sigma}\Delta\mathbf{H}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=-\frac{\nabla p}{\rho}-\frac{1}{4\pi\rho}\mathbf{H}\times\operatorname{rot}\mathbf{H}+\frac{\eta}{\rho}\Delta\mathbf{v}+\frac{1}{\rho}\left(\frac{\eta}{3}+\zeta\right)\Delta\mathbf{v}$$

$$p=p\,(\rho,\ T)$$

(v is the velocity, ρ is the density of the fluid, η, ξ are the two viscosity coefficients of the fluid, p is pressure, H is the magnetic-field intensity, σ is the conductivity, T is temperature, c is the velocity of light).

In the flow considered below the dissipative coefficients shall be considered constant; however, the heat-transfer equations can be reduced to the entropy-conservation equation (the conditions for adiabaticity of motion) (see [1], §52).

Let us consider an infinite plane layer having a thickness 2a, which is bounded by planar solid walls and is filled with a conducting fluid. A beam of waves having a width 2b propagates along the axis of the layer. The end faces of the channel are closed with films which are transparent to the waves. A uniform magnetic field having an intensity H is applied perpendicular to the planes which bound the channel. Let us determine the flow which develops in such a system.

Later on it will be required to determine a certain relationship for the magnetohydrodynamic waves – in particular, the absorption coefficient α. The x axis shall be directed along the channel axis, while the y axis shall be directed perpendicular to its walls. We shall consider a wave of the form

$$v=v_0\cos(\omega t-\mathbf{kx})\,e^{-\alpha x} \tag{2}$$

where v_0 is the velocity amplitude, w is the frequency, k is the wave vector, the phase velocity of the wave

$$u=\omega\,/\,|k| \tag{3}$$

and α is the absorption coefficient, with

$$\alpha=\langle Q\rangle\,/\,2\,\langle q\rangle, \tag{4}$$

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⟨Q⟩ being the time-averaged quantity of energy dissipated per second per cubic meter; ⟨q⟩ is the average energy density in the wave (the absorption coefficient α is assumed small, $\alpha L \ll 1$, where L is the channel length).

In the case of the transverse magnetic field considered, the propagation velocities of the magnetohydrodynamic waves and their limiting values are determined, for example, in [1].

The quantities ⟨Q⟩ and ⟨q⟩ are determined by the following expressions:

$$\langle Q\rangle = \frac{k^2 v_0^2}{2}\left[\frac{4}{3}\eta + \zeta\right] + \frac{H_0^2 c^2}{4\pi^2 \sigma u_0}, \qquad \langle q\rangle = \frac{\rho v_0^2 u_0}{2}\left\{1 + \frac{H_0^2}{4\pi\rho u_0^2}\right\}. \tag{5}$$

For the absorption coefficients we obtain the expressions when

$$H^2 \ll 4\pi\rho u_0^2, \quad \alpha = \frac{\omega^2}{\rho u_0^3}\left[\left(\frac{4}{3}\eta + \zeta\right) + \frac{H_0^2 c^2}{4\pi^2 \sigma u_0^2}\right], \tag{6}$$

and when

$$H^2 \gg 4\pi\rho u_0^2, \quad \alpha = \frac{2\pi\omega^2}{u_0^2 H^2}\left[\left(\frac{4}{3}\eta + \zeta\right) + \frac{H_0^2 c^2}{4\pi^2 \sigma u_0^2}\right]. \tag{7}$$

The velocity v in the wave is associated with a small correction h_y to the field by the relationships: for the condition (6)

$$h_y \approx v_x H_0 / u_0; \tag{8}$$

for the condition (7)

$$v_x \approx \frac{h_y}{\sqrt{4\pi\rho}}, \quad u_0 = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2}. \tag{9}$$

Here u_0 is the propagation velocity of sound in the medium.

For Alfvén waves the absorption coefficient is equal to [1]

$$\alpha = \frac{\omega^2}{2u_1^3}\left(\frac{\eta}{\rho} + \frac{c^2}{4\pi\sigma}\right). \tag{10}$$

We shall now seek the solution of the system (1) in the form

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}_1 + \mathbf{h}_2 + \ldots, \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots, \quad \rho = \rho_0 + \rho_1 + \rho_2 + \ldots, \tag{11}$$

the quantities v_1 and h_1 being magnetohydrodynamic waves.

The quantities $\mathbf{h}_2$, ρ_2, $\mathbf{v}_2$ represent the next approximation in the solution of the system (1); under these conditions a term which is independent of time – the flow velocity – appears besides the time-dependent term.

Let us write out the equations for the second approximation for the time-averaged values $\langle \mathbf{v}_2\rangle$ and $\langle \mathbf{h}_2\rangle$ of the quantities $\mathbf{v}_2$ and $\mathbf{h}_2$; for this purpose we substitute the expansion (11) into the system (1), consider terms of up to the second approximation inclusive, and average the resulting equations with respect to time (the so-called Schrödinger method; see, for example [2]). As a result we obtain

$$\begin{gathered}\operatorname{div}\langle \mathbf{v}_2\rangle = 0\\ \frac{c^2}{4\pi\sigma}\Delta\langle \mathbf{h}_2\rangle + \operatorname{rot}[\langle \mathbf{v}_2 \mathbf{H}_0\rangle] + \operatorname{rot}[\langle \mathbf{v}_1 \mathbf{h}_1\rangle] = 0\\ \operatorname{div}\langle \mathbf{h}_2\rangle = 0\end{gathered} \tag{12}$$

$$\operatorname{rot}\langle(\mathbf{v}_1\nabla)\mathbf{v}_1\rangle = \nu \operatorname{rot}\Delta\langle \mathbf{v}_2\rangle - \frac{1}{4\pi\rho_0}\operatorname{rot}\langle[\mathbf{H}_0 \operatorname{rot}\mathbf{h}_2]\rangle - \frac{1}{4\pi\rho_0}\operatorname{rot}\langle[\mathbf{h}_1 \operatorname{rot}\mathbf{h}_1]\rangle, \quad \nu = \eta/\rho.$$

The curl operation is applied to the last equation of the system after averaging has been performed.

We will be interested in flows which develop due to the absorption of the momentum of a wave in the middle portion of the channel only, neglecting the effect of the ends of the channel; we shall likewise assume that the flow velocity is directed along the x axis throughout, while the variation of all of the quantities associated with it takes place considerably more rapidly along the y axis than along the x axis (i.e., we neglect all quantities of the form $\partial/\partial x$ in comparison with $\partial/\partial y$).

Under these assumptions we obtain the following system of equations [starting from Eqs. (12)] for the flows in the middle portion of the channel:

$$\frac{c^2}{4\pi\sigma}\frac{\partial^2 \langle h_{2x}\rangle}{\partial y^2} + H_0 \frac{\partial \langle v_{2x}\rangle}{\partial y} = 0 \tag{13}$$

$$\left\langle v_{1x}\frac{\partial v_{1x}}{\partial x}\right\rangle = \nu \frac{\partial^2 \langle v_{2x}\rangle}{\partial y^2} + \frac{H_0}{4\pi\rho}\frac{\partial \langle h_{2x}\rangle}{\partial y} + C$$

where C is the integration constant.

The boundary conditions for the system (13) have the form

$$h_{2x} = 0 \quad \text{for} \quad y = \pm a, \quad \langle v_{2x}\rangle = 0 \quad \text{for} \quad y = \pm a \tag{14}$$

From the system (13) one can easily obtain the equation for the quantity $\langle v_{2X}\rangle$, which has the form

$$\frac{\partial^2 \langle v_{2x}\rangle}{\partial y^2} - D^2 \langle v_{2x}\rangle + c_1 = -\frac{\alpha v_0^2 \theta(y)}{2\nu} \tag{15}$$

$$D^2 = \frac{H_0^2 \sigma}{c^2 \eta}, \quad \theta(y) = \begin{cases} 0 & (b < |y| \leqslant a) \\ 1 & (0 \leqslant |y| \leqslant b) \end{cases} \tag{16}$$

The solution of Eq. (15) with the boundary conditions (14) has the form

$$\langle v_{2x}\rangle = \begin{cases} a_1(\operatorname{ch} Dy - \operatorname{ch} Da) + a_2(\operatorname{ch} Db - \operatorname{ch} Dy) & (|y| < b) \\ a_1(\operatorname{ch} Dy - \operatorname{ch} Da) \quad (b < |y| < a) & a_2 = \alpha v_0^2 / 2\nu D^2 \end{cases} \tag{17}$$

where a_1 is a constant which is determined from the condition of mass conservation over the channel cross section:

$$\rho_0 \int_0^a \langle v_{2x}\rangle \, dy = 0 \tag{18}$$

Substituting Eq. (17) into this equation, we obtain the following expression for a_1:

$$a_1 = \frac{a_2}{\operatorname{ch} Db}\left\{\frac{b \operatorname{ch} Db - (\operatorname{sh} Db)/D}{a \operatorname{ch} Da - (\operatorname{sh} Da)/D}\right\}. \tag{19}$$

As has already been noted, the presence of a magnetic field will affect both the absolute values of the flow velocities and the configuration of the velocity profile. The degree to which the magnetic field affects the flow velocity depends on the quantity Da.

Thus, when $Da \ll 1$, we obtain

$$\langle v_{2x}\rangle = \begin{cases} {}^1\!/_2\, a_2 b^2 D^2 [(1 - b/a) + (y/b)^2 (b^3/a^3 - 1)] & (|y| < b) \\ {}^1\!/_4\, \alpha v_0^2 b^3 (y^2 - a^2) \nu^{-1} a^{-3} & (b < |y| < a) \end{cases} \tag{20}$$

Thus, in this case the configuration of the velocity profile of the flow caused by the wave is the same as it is in conventional acoustic flow. An increase in the absolute value of the velocity may occur only at the expense of increasing the absorption coefficient (other conditions remaining equal). The ratio between the maximal values of the flow velocities in the conventional $\langle v_{2X}\rangle_1$ and magnetohydrodynamic $\langle v_{2X}\rangle_2$ cases is equal to

$$\frac{\langle v_{2x}\rangle_2}{\langle v_{2x}\rangle_1} = \frac{\alpha}{\alpha_1} = \frac{(4/3)\eta + \zeta + H_0^2 c^2 / u_0^2 4\pi\sigma}{(4/3)\eta + \zeta} \approx 1 + \frac{H_0^2 c^2}{\eta u_0^2 4\pi^2 b} \tag{21}$$

In the case when $Da \gg 1$, we have

$$\langle v_{2x}\rangle = \begin{cases} {}^1\!/_2\, a_2 (b/a)[e^{-D(a-|y|)} - 1) + (a/b)(1 - e^{-D(b-|y|)})] & (|y| < b) \\ {}^1\!/_2\, a_2 a^{-1}[1 - e^{-D(a-|y|)}] \quad (b < |y| < a) \end{cases} \tag{22}$$

The configuration of the velocity profile becomes flatter under these conditions.

Let us emphasize the fact that the expressions obtained above for the stream velocities are valid only for the following constraints:

$$\alpha L \leqslant 1, \quad H^2 \leqslant 4\pi\rho v_0^2, \quad 2\pi v_0 u_0 / \omega\nu \leqslant 1 \tag{23}$$

The latter constraint – a small Reynolds number – is necessary for a series of the type (11) to converge.

Let us now consider the case when under the conditions (23) the second inequality is replaced by the inverse inequality, while the remaining two are retained [i.e., $H^2 \gg 4\pi \rho u_0^2$; see (7)]. Since in this case the quantity h_{1y} is independent of y, it follows that the system of equations (12) for the second-approximation quantity does not change, and consequently the equation for the flow velocity remains unchanged. The boundary conditions likewise remain unchanged, and therefore the form of the solution remains the same; only the absorption coefficient and the propagation velocity of the wave itself change. Under conditions (9) the ratio between the maximal flow velocities in the conventional $\langle v_{2x}\rangle_1$ and magnetohydrodynamic $\langle v_{2x}\rangle_2$ cases is equal to

$$\frac{\langle v_{2x}\rangle_2}{\langle v_{2x}\rangle_1} = \frac{2\pi u_0^2}{H_0^2}\left\{1 + \frac{H_0^2 c^2}{4\pi\sigma u_0^2 (4\eta/3 + \zeta)}\right\} \tag{24}$$

For practical conducting fluids, plasma in particular, this ratio may be a quantity of the order of 10^2–10^3.

In conclusion let us note the following. The existence of a constant flow leads to the appearance of time-constant electric fields directed along the z axis and having an intensity $|E| = j_z\sigma$, where j_z is the z component of the current density.

Using the equation $j = c\ \mathrm{curl}\ H/4\pi$ and Eq. (13), we obtain $E = \sigma^2 H_0 \langle v_{2x}\rangle / c$.

The quantity E may have a value of the order of 10^{-1}–10^{-2} μV/cm for flows in electrolytes, for example, if one is able to create a flow having a velocity of 10^2 cm/sec in a KCl solution having a conductivity $\sim 0.1\ \Omega^{-1} \cdot \mathrm{cm}^{-1}$ at a magnetic-field intensity H = 3 kG.

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